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Additive Invariant Functionals for Dynamical Systems

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We consider the problem of defining completely a class of additive conservation laws for the generalized Liouville equation whose characteristics are given by an arbitrary system of first-order ordinary differential equations. We first show that if the conservation law, a time-invariant functional, is additive on functions having disjoint compact support in phase space, then it is represented by an integral over phase space of a kernel which is a function of the solution to the Liouville equation. Then we use the fact that in classical mechanics phase space is usually a direct product of physical space and velocity space (Newtonian systems). We prove that for such systems the aforementioned representation of the invariant functionals will hold for conservation laws which are additive only in physical space; i.e., additivity in physical space automatically implies additivity in the whole phase space. We extend the results to include non-degenerate Hamiltonian systems, and, more generally, to include both conservative and dissipative dynamical systems. Some applications of the results are discussed.

KEY WORDS: Kinetic equations; dynamical systems; Liouville equation; conservation laws; nonlinear functionals.

1. INTRODUCTION

Let $f(x, v, t)$ be a distribution function of particles at time t, having velocity $v \in \mathbb{R}^n$, at position $x \in \mathbb{R}^n$, satisfying the following nonlinear kinetic equation,

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + F \cdot \frac{\partial f}{\partial v} = \mathbf{S}(f), \qquad t > 0 \tag{1}$$

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where $F(x, v, t)$ is an external (conservative) force and $\mathbf{S}(f)$ denotes a nonlinear operator which describes interactions between particles. This could be any type collisional integral (e.g., a Boltzmann or a Fokker–Planck–Landau collision operator) or a nonlinear Vlasov force term, or a sum of such operators. The notations $\partial/\partial x$ and $\partial/\partial v$ are the gradients, respectively, with respect to the positional variable x and the velocity variable v.

A classical problem for the above kinetic equation (1) is to describe all possible conservation laws, i.e., functionals $G(f)$, which are preserved (in time) for any solution $f(x, v, t)$. If the operator $\mathbf{S}(f)$ has some dissipative properties (e.g., the Boltzmann or Fokker–Planck–Landau case), then functionals with monotone behavior (i.e., "H-theorems") are also of great importance. In both settings of conservation laws or dissipative functionals, there is the limiting case of a very dilute gas, for example, when one can neglect particle interactions and set $\mathbf{S}(f) \equiv 0$. Then conservation laws $G(f)$ become the object of interest for the linear Liouville equation

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + F \cdot \frac{\partial f}{\partial v} = 0 \tag{2}$$

We also note that there exist infinitely many conservation laws for the spatially homogeneous Boltzmann equation for Maxwell molecules[1] (a completely integrable system); however they can not be generalized to the full Boltzmann equation just because of properties of the Liouville operator. On the other hand, a knowledge of a complete set of invariants for the Liouville equation makes it possible to reduce the similar problem for the nonlinear kinetic equation to its spatially homogeneous version which is always much simpler. Conservation laws reflect the most fundamental properties of physical systems and constitute a basis for the mathematical study of such systems. Therefore, it is important to be able to describe a complete set (in a certain class) of invariant functionals $G(f)$ of (2).

In the most general case, our problem is the following: we consider an "arbitrary" dynamical system (i.e., a nonautonomous ordinary differential equation (ODE) in $\mathbb{R}^m$, $m = 1,...$):

$$\frac{dz}{dt} = v(z; t), \qquad z \in \mathbb{R}^m \tag{3}$$

and the corresponding "pseudo-Liouville" equation

$$\left(\frac{\partial}{\partial t} + v(z; t) \frac{\partial}{\partial z}\right) f(z, t) = 0 \tag{4}$$

(The term "pseudo" is used to stress that the phase space Lebesgue measure is not necessarily preserved in the general case.) Let $G_t(f)$ be a one parameter family of functionals (acting on the z-variable) such that $G_t[f(\cdot, t)] = const.$ if $f(z, t)$ satisfies (4). It is obvious that (4) has an infinite set of conservation laws, as any function $\Psi(f)$ formally satisfies (4). For the "conservative" case (i.e., $div_{(X, V)}(V, F) = 0$) for (2), a typical conservation law is

$$G_t(f) = \int_{\mathbb{R}^n \times \mathbb{R}^n} dx\, dv\, \Psi[f(x, v, t)] = const. \tag{5}$$

It is also clear how to construct the corresponding (time-dependent) functional $G_t(f)$ for the general case (4). But it is unclear how to define uniquely classes of conservation laws for (4) having the form of (5). Roughly speaking, the problem at hand is to ascertain how to obtain certain minimal conditions on a class $\mathscr{A}$ of functionals such that all conservation laws in this class have the form of (5). The main goal of this work is to prove that in many physically relevant cases (non-degenerate Hamiltonian systems, etc.), the only natural condition of additivity of functionals $G_t(f)$ in physical space (i.e., with respect to the positional variable x in (2)) allows us to define completely a set of conservation laws having the form of (5).

To the best of our knowledge, the only previous result of this type was obtained by V. V. Vedenyapin[5] (in connection with the problem of uniqueness of the Boltzmann H-function) for the case of free motion (i.e., $F \equiv 0$ in (2)),

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = 0, \qquad x, v \in \mathbb{R}^3$$

under very strong restrictions on the functionals $G(f)$, requiring G to be twice continuously differentiable. Even in such a trivial case, the proof that all time-independent spatially additive conservation laws can be expressed as

$$G(f) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx\, dv\, \Psi(f, v)$$

with a certain function $\Psi(u, v)$, is surprisingly complicated and cannot apparently be generalized to the nontrivial case (3). Our approach differs completely from ref. 5 and makes it possible to solve the problem in the most general case.

It is clear that one can discuss such problems in terms of functionals on the solution of Liouville-type equations (typical for kinetic theory) or in terms of functions of phase sets (typical for the theory of dynamical systems and for ergodic theory). We prefer the first way since it makes possible a direct application of results to nonlinear kinetic equations. However some of results of the paper can be easily formulated in terms of ergodic theory.[4] In particular, it follows from our results that, roughly speaking, the following fact is valid for many physically relevant dynamical systems: any invariant (in time) continuous function of phase sets, which is assumed to have additive properties only with respect to the spatial variable x, appears to be an invariant measure on the whole phase space. Because of length considerations for the present work, we shall discuss other possible applications in a separate paper.

The paper is organized as follows. In Sections 2 and 3, we prove some auxiliary results (Lemmas 1 and 2) concerning general properties of additive nonlinear functionals in spaces of integrable functions. In Section 4, we consider general dynamical systems (3) and show that the most general conditions of additivity in phase space are sufficient to define uniquely a set of invariant functionals. This constitutes Theorem 1. In Section 5, we study a class of Newtonian dynamical systems for which phase space is a Cartesian product of physical (position) space $\mathbf{X}$ and velocity space $\mathbf{V}$. (Generally speaking, $\mathbf{V}$ is the tangent bundle to $\mathbf{X}$, but we consider the simplest case $\mathbf{X} = \mathbb{R}^n$, $\mathbf{V} = \mathbb{R}^n$.) We define a notion of semi-additivity (i.e., additivity with respect to $\mathbf{X}$ only) of functionals in Section 5 and prove in Section 6 our main result (Theorem 2). This describes all spatially additive conservation laws for Newtonian systems and its proof is based on a geometric result (Lemma 4) which shows that semi-additivity implies full (phase space) additivity in the Newtonian case. Hamiltonian and Lagrangian systems are considered in Section 7, and some generalizations and applications are discussed in Sections 8 and 9.

Our main goal in this paper is to study some classes of functionals on the solution of the PDE (2), not to investigate in detail the solution of the ODE (3). Therefore, we shall usually assume without proof that there exists a certain time interval $[0, T]$ such that the initial value problem for (3) with any initial condition $z(0) = z_0 \in \mathbb{R}^m$ has a unique solution $z(t) \in C_1[0, T]$; moreover the Jacobian $|Dz(t)/D(z)|$ has no zeroes. This general assumption is sometimes omitted for brevity in the formulation of the theorems. Moreover, all subsets $\Delta \subset \mathbb{R}^m$ which we consider below are assumed bounded and measurable although this condition is not always mentioned.

2. ADDITIVE FUNCTIONALS

We consider a set of nonnegative functions $f(z)$, $z \in \mathbb{R}^m$ $(m = 1, 2, ...)$, and denote $L_p^{+,c} = \{f(z): \mathbb{R}^m \to \mathbb{R}_+ : f$ has compact support,

$$\|f\|^p = \int_{\mathbb{R}^m} dz\, f^p(z) < \infty\} \tag{6}$$

where $1 \leqslant p < \infty$ is a fixed number.

Let $G(f)$: $L_p^{+,c} \to \mathbb{R}$ be a real functional and assume that:

(a) $G(f)$ is continuous with respect to L_p on a certain set $\mathscr{D}(G) \subset L_p^{+,c}$;

(b) If $f_1, f_2 \in \mathscr{D}(G)$ and $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \varnothing$, then

$$G(f_1 + f_2) = G(f_1) + G(f_2) \tag{7}$$

(c) For any number $0 \leqslant u < \infty$, and for any bounded measurable set $\Delta \subset \mathbb{R}^m$, the function $f(z) = u\mathscr{X}_\Delta(z) \in \mathscr{D}(G)$ where $\mathscr{X}_\Delta(\cdot)$ is the characteristic function of Δ.

Definition 1. A functional $G(\cdot)$: $L_p^{+,c} \to \mathbb{R}$ is called a functional class of $\mathscr{A}$ (i.e., "additive") if $G(f)$, $f \in L_p^{+,c}$, satisfies conditions (a), (b), and (c). In such a case, we say $G(\cdot) \in \mathscr{A}$.

It is easy to prove the following:

Lemma 1. If $G(\cdot) \in \mathscr{A}$, then $G(f)$ can be defined for all $f \in \mathscr{D}(G)$ by a certain function $g(u; z)$: $\mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$. The function $g(u; z)$ has the following properties:

(1) $g(0; z) = 0$;

(2) For any fixed $u > 0$, $g(u; z) \in L_1^{loc}(\mathbb{R}^m)$;

(3) For any fixed $\Delta \subset \mathbb{R}^m$ of finite Lebesgue measure,

$$e(u; \Delta) := \int_\Delta dz\, g(u; z) \tag{8}$$

is continuous for all $u \in [0, \infty)$. Moreover, for all such simple, finite-valued, nonnegative functions f,

$$G(f) = \int_{\mathbb{R}^m} dz\, g(f(z), z) \tag{9}$$

(Cf., e.g., the discussion in Remark 4 concluding this section).

Proof. We obtain $g(u; z)$ by a Radon–Nikodym argument. Observe that simple nonnegative functions

$$\hat{f}(z) = \sum_{k=1}^{N} u_k \mathscr{X}_{\Delta_k}(z), \qquad u_k \geqslant 0, \quad \Delta_i \cap \Delta_j = \phi, \quad i \neq j \tag{10}$$

form a dense set in $L_p^{+,c}(\mathbb{R}^m)$. Additionally, all such functions are contained in the domain $\mathscr{D}(G)$ because of assumptions (b) and (c). Because of assumption (a), $G(f)$ is continuous on $\mathscr{D}(G)$, and so it is sufficient to show Lemma 1 for simple nonnegative functions.

By observing that

$$f_1(z) \equiv 0, \qquad f_2(z) = u\mathscr{X}_\Delta(z), \qquad \text{meas } \Delta = 0 \tag{11}$$

are equivalent in $L_p(\mathbb{R}^m)$, we derive from assumption (b) that $G(0) = 0$. Now fix $u > 0$ and consider

$$e(u; \Delta) = G(u\mathscr{X}_\Delta(z)) \tag{12}$$

From assumption (b), we can conclude

$$e(u; \Delta_1 \cup \Delta_2) = e(u; \Delta_1) + e(u; \Delta_2), \qquad \Delta_1 \cap \Delta_2 = \varnothing \tag{13}$$

We can moreover conclude that the function $e(u; \Delta)$ is σ-additive; let $A = U_{k=1}^{\infty} \Delta_k$, with $\Delta_i \cap \Delta_j = \varnothing$, $i \neq j$, and meas $(A) < \infty$. Then

$$e(u; A) = \sum_{k=1}^{\infty} e(u; \Delta_k) \tag{14}$$

since the function $u\mathscr{X}_A(z) \in \mathscr{D}(G)$. Thus the function $e(u; \Delta)$, for any fixed $u > 0$ is a measure, not necessarily positive on $\mathbb{R}^m$. Moreover, $e(u; \Delta) = 0$ if meas $\Delta = 0$. Thus, we can appeal to the Radon–Nikodym theorem and deduce that there exists a locally integrable function $g(u; z)$ defined on $\mathbb{R}^m$ such that

$$e(u; \Delta) = \int_\Delta dz\, g(u; z) \tag{15}$$

For any fixed set Δ, the function $e(u; \Delta)$ is obviously continuous for all $u \geqslant 0$, because of the continuity of $G(\cdot)$.

Let $\hat{f}(z)$ be a simple nonnegative function (10); then

$$G(\hat{f}) = \sum_{k=1}^{N} e(u_k; \Delta_k) = \sum_{k=1}^{N} \int_{\Delta_k} dz\, g(u_k; z) \tag{16}$$

We can rewrite this equality as

$$G(\hat{f}) = \int_{\mathbb{R}^m} dz\, g[\hat{f}(z); z] \tag{17}$$

because of (13)–(15), where $\hat{f}(z)$ is any (finite-valued) simple nonnegative function. For any $f \in \mathscr{D}(G)$, we can construct an approximating sequence $\{f_n\}$ of simple functions and obtain by continuity,

$$G(f) = \lim_{n \to \infty} \int_{\mathbb{R}^m} g[\hat{f}_n(z); z] \tag{18}$$

This completes the proof.

Remark 1. We have assumed from the very beginning that $f(z)$ is nonnegative, although we did not use this condition in the proof of Lemma 1. A reason for restricting our study to nonnegative functions is that $f(z, t)$ solving (1) and (2) (with $z = (x, v)$) is a probability density. A typical functional in kinetic theory is the Boltzmann H-functional [2, pp. 97] which corresponds to $g(u; z) = u \ln u$ in (8), so that $g(u; z)$ is often defined only for nonnegative u. The important physical restriction on f is that $f \in L_1$ for most applications. On the other hand, nothing changes in Lemma 1 and in all the subsequent results of Sections 3–8 if we replace the condition $f(z) \geqslant 0$ by the condition $a \leqslant f(z) \leqslant b$ or even $|f(z)| < \infty$ for almost all $z \in \mathbb{R}^m$. Applications will be considered in the concluding paragraphs of Section 8. Our goal is to describe the widest possible class of functionals satisfying Lemma 1, which include those defined on solutions to (1).

Remark 2. The above conditions (a), (b), and (c) are of course valid for bounded linear functionals in $L_p(\mathbb{R}^m)$. In this case, we can deduce that $g(u; z) = uh(z)$ with $h \in L_q$, $q = p/(p-1)$ because of the duality $(L_p)^* = L_q$, $1 \leqslant p < \infty$. Roughly speaking, any linear functional on L_p is identified with a certain L_q-function $h(z)$ of m variables, whereas any additive functional $G(\cdot) \in \mathscr{A}$ is represented by a function $g(u; z)$ of $(m+1)$ variables.

Remark 3. Lemma 1 is not true for $p = \infty$. This is clear for the case of linear functionals, since the conjugate space L_∞^* is the space of finitely additive measures. The L_p spaces, $1 \leqslant p < \infty$, are sufficient for our purposes and we do not pursue the generalization of Lemma 1 to the case $p = \infty$ (Indeed, observe that the proof of σ-additivity (14) fails in this case).

Remark 4. Under certain conditions on $g(u; z)$, we can actually carry out an analysis based on the Lebesgue Dominated Convergence Theorem and approach the limiting function under the integral sign in (18). In this setting, (17) is valid for any $f \in \mathscr{D}(G)$. Moreover, a knowledge of the behavior of $g(u; z)$ with respect to $u \in \mathbb{R}^+$, $z \in \mathbb{R}^m$ enables one to characterize more precisely the domain $\mathscr{D}(G)$ of continuity of the functional. For example, if $g(u; z)$ is continuous with respect to u for almost all $z \in \mathbb{R}^m$, and measurable with respect to z for all values of u, then it is well known [3, pp. 20–27] that such Caratheodory conditions on $g(u; z)$ imply $\mathscr{D}(G) = L_p(\mathbb{R}^m)$. For most of the analysis in this work, we shall only need the condition on g discussed in Lemma 1.

3. TRANSFORMATIONS OF FUNCTIONS AND FUNCTIONALS

For brevity, we introduce

Definition 2. The function $g(u; z)$ defined in Lemma 1 will be called the kernel of the functional $G(\cdot) \in \mathscr{A}$.

Let us consider a diffeomorphism $\Psi\colon \mathbb{R}^m \to \mathbb{R}^m$, $\Psi(\cdot) \in C_1(\mathbb{R}^m)$ and that the Jacobian determinant

$$\mathscr{J}\,|\Psi(z)| = |D(\Psi(z))/D(z)| \neq 0, \qquad z \in \mathbb{R}^m \tag{19}$$

We denote

$$y = \Psi(z) \Leftrightarrow z = \Psi^{-1}(z), \qquad y, z \in \mathbb{R}^m \tag{20}$$

and assume that the function $f(z)$ is transformed under the diffeomorphism Ψ to a new function $f_1(\cdot)$ given by

$$f_1(z) = f[\Psi^{-1}(z)] \tag{21}$$

In particular, if $f(z) = u\mathscr{X}_\Delta(z)$, then

$$f_1(z) = u\mathscr{X}_\Delta[\Psi^{-1}(z)] = u\mathscr{X}_{\Psi(\Delta)}(z) \tag{22}$$

The transformation of the functional $G(f)$ is defined in a natural manner, i.e.,

$$G_1(f_1) = G(f) \tag{23}$$

It is clear that if $G(\cdot) \in \mathscr{A}$, then $G_1(\cdot) \in \mathscr{A}$. We denote the kernel of G_1 by $g_1(u; z)$ and obtain from (22) and (23),

$$G_1[u\mathscr{X}_{\Psi(\Delta)}] = \int_{\Psi(\Delta)} dy\, g_1(u; y)$$

$$= \int_\Delta dz\, \mathscr{J}[\Psi(z) \mid z]\, g_1(u; \Psi(z)) = G[u\chi_\Delta]$$

$$= \int_\Delta dz\, g(u; z)$$

Thus, for Δ an arbitrary measurable set

$$g_1(u; z) = \mathscr{J}[\Psi^{-1}(z) \mid z]\, g(u; \Psi^{-1}(z)) \tag{24}$$

We have shown:

Lemma 2. The equalities

$$y = \Psi(z), \qquad f_1(y) = f(z), \qquad G_1(f_1) = G(f) \tag{25}$$

with (21), define uniquely the kernel $g_1(u; z)$ of $G_1(f)$ by (24). Therefore,

$$G_1[u\chi_\Delta] = e_1(u; \Delta) = \int_\Delta dz \left| \frac{D[\Psi^{-1}(z)]}{D(z)} \right| g(u; \Psi^{-1}(z)) \tag{26}$$

In the next section, we consider some applications to dynamical systems.

4. DYNAMICAL SYSTEMS AND INVARIANT FUNCTIONALS

We consider a dynamical system, i.e., a system of ODE's,

$$\frac{dz}{dt} = a(z, t), \qquad z \in \mathbb{R}^m, \quad t \in \mathbb{R}, \quad a\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \tag{27}$$

(where t plays the role of a time variable). We assume that $a(z, t) \in C_1(\mathbb{R}^m \times \mathbb{R})$ and that the Cauchy problem with any initial data

$$z(0) = z_0 \in \mathbb{R}^m \tag{28}$$

has a unique solution on the interval $0 \leqslant t \leqslant T$, with certain $T > 0$. The solution $z(t)$ can be written as

$$z(t) = \hat{\mathbf{S}}(t)\, z_0, \quad 0 \leqslant t \leqslant T, \qquad \hat{\mathbf{S}}(0) = I \tag{29}$$

where $\hat{\mathbf{S}}(t)$ denotes a one-parameter family of transformations $\hat{\mathbf{S}}$: $\mathbb{R}^m \to \mathbb{R}^m$. Moreover, we assume that the Jacobian has no zeroes, i.e.,

$$\left| \frac{D[\hat{\mathbf{S}}(t)\, z]}{D(z)} \right| \neq 0, \qquad z \in \mathbb{R}^m, \quad 0 \leqslant t \leqslant T \tag{30}$$

Then the transformations $\hat{\mathbf{S}}(t)$ are invertible and the inverse transformations

$$z_0 = \hat{\mathbf{S}}^{-1}(t)\, z \tag{31}$$

are defined for all $z \in \mathbb{R}^m$, $t \in [0, T]$.

We associate with (27) the first order PDE for a function $f(z, t)$

$$\frac{\partial f}{\partial t} + a(z, t) \cdot \frac{\partial f}{\partial z} = 0 \tag{32}$$

It follows from the aforementioned properties of the characteristic dynamical system (27) that the solution of (32) satisfying the initial conditions

$$f(z, 0) = f_0(z) \tag{33}$$

can be expressed as

$$f(z, t) = f_0[\hat{\mathbf{S}}^{-1}(t)\, z], \qquad 0 \leqslant t \leqslant T.$$

If we interpret $f_0(z_0)$ as a probability density,

$$\int_{\mathbb{R}^m} dz_0\, f_0(z_0) = 1 \tag{34}$$

of initial conditions (28) for the system (27), then the corresponding probability density $p(z, t)$ for $t > 0$ is given by the formula

$$p(z, t) = |D[\hat{\mathbf{S}}^{-1}(t)\, z]/D(z)|\, f(z, t) = \left| \frac{D[\hat{\mathbf{S}}^{-1}(t)\, z]}{D(z)} \right| f_0(\hat{\mathbf{S}}^{-1}(t)\, z). \tag{35}$$

It is clear that the equality

$$\int_{\mathbb{R}^m} dz\, p(z, t) = \int_{\mathbb{R}^m} dz\, p(z, 0) = 1 \tag{36}$$

expresses the mass conservation law for the dynamical system (27), or for (32). The integral on the left hand side of (36) with $p(z, t)$ given by (35) can be regarded as a time-dependent linear functional, given by

$$G_t[f(\cdot, t)] = \int_{\mathbb{R}^m} dz \left| \frac{D[\hat{\mathbf{S}}^{-1}(t)\, z]}{D(z)} \right| f(z, t) = const.$$

on the solution of (32). To be more precise, we define a one parameter family of linear functions

$$G_t(f) = \int_{\mathbb{R}^m} dz \left| \frac{D[\hat{\mathbf{S}}^{-1}(t)\, z]}{D(z)} \right| f(z) \tag{37}$$

and prove that

$$\frac{d}{dt} G_t[f(\cdot, t)] = 0,$$

or, in other words, that the functional $G_t(f)$ is conserved on solutions $f(z, t)$ of (32). Such functionals will be called the invariant functionals (or conservation laws) for (32).

It is obvious that, for any function $\Psi(u; z)$: $\mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ with certain additional integrability conditions, the functional

$$G_t(f) = \int_{\mathbb{R}^m} dz \left| \frac{D[\hat{\mathbf{S}}^{-1}(t)\, z]}{D(z)} \right| \Psi\{f(z); \hat{\mathbf{S}}^{-1}(t)\, z\} \tag{38}$$

is also conserved for solutions of (32).

It is less trivial, however, to define a class of functionals in which all conservation laws are similar to (38), i.e., any invariant functional is defined by certain functions of $(m+1)$ variables, $\Psi(u; z)$. By making use of Lemmas 1 and 2, we are now able to prove the following:

Theorem 1. Any conservation law for (32) expressed by

$$\frac{d}{dt} G_t(f) = 0 \tag{39}$$

where $G_t(\cdot) \in \mathscr{A}$, $0 \leqslant t \leqslant T$, is completely defined by a certain function $g(u; z)$ of $(m+1)$ variables. The kernel $g_t(u; z)$ of G_t can be written as

$$g_t(u; z) = \left| \frac{D[\hat{\mathbf{S}}^{-1}(t)\, z]}{D(z)} \right| g(u; \hat{\mathbf{S}}^{-1}(t)\, z) \tag{40}$$

Proof. We fix $0 < t \leqslant T$, and set

$$\begin{gathered} y = \hat{\mathbf{S}}(t)\, z = \Psi(z), \qquad f_1(z) = f[\hat{\mathbf{S}}^{-1}(t)\, z] = f(z, t) \\ G_1(f_1) = G_t(f) \end{gathered} \tag{41}$$

Apply now Lemma 2 since all equalities (25) are satisfied. Then we obtain from (26) the kernel $g_1(u; z) = g_t(u; z)$, and express $g_1(u; z)$ in the notation in (41). The results in formula (40) and completes the proof.

Remark. Theorem 1 is almost obvious. However, as for many "obvious" theorems, Theorem 1 requires rather technical machinery. In Sections 5 and 6, we show a stronger result which is much less obvious and valid for the important special case of Newtonian systems.

5. NEWTONIAN DYNAMICAL SYSTEMS: POSITIONS AND VELOCITIES

In Section 4, we treated the most general case of a dynamical system (27) in $\mathbb{R}^m$. In this case, all m coordinates $\{z_1, z_2, ..., z_m\}$ of the phase point $z \in \mathbb{R}^m$ are equally weighted and we do not distinguish between them. However, the following situation is typical for dynamical systems of Newtonian mechanics: $m = 2n$ $(n = 1, 2, ...)$, and the phase point z is a pair

$$z = (x, v), \qquad x \in \mathbb{R}^n, \quad v \in \mathbb{R}^n \tag{42}$$

of position x and velocity v. In such a case, the dynamical system (27) becomes

$$\begin{gathered} \frac{dx}{dt} = v, \qquad \frac{dv}{dt} = w(x, v, t), \qquad x, v \in \mathbb{R}^n \\ w\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to R^n \end{gathered} \tag{43}$$

Remark. Special cases of (nondegenerate) Lagrangian and Hamiltonian systems will be considered below. Roughly speaking, these systems are very similar to (42).

We stress at this juncture that the phase space of (43) has a special structure: it is a Cartesian product of a set of positions $\mathbf{X}$ (physical space) and a set **V** of vectors tangent to **X** (velocity space). Usually this distribution does not play a role if we consider the simplest case $\mathbf{X} = \mathbb{R}^n$, $\mathbf{V} = \mathbb{R}^n$. However, as we shall see shortly, the difference between x and v is important for our problem.

In Section 2, we introduced a class $\mathscr{A}$ of additive functionals and showed that each such functional can be defined by its kernel $g(u; z)$. Besides the usual continuity assumption, the main assumption was b (7) of full additivity in phase space. However, in physics, we often have many reasons to assume additivity in physical space **X** and much less reason for additivity in the entire phase space.

In order to better appreciate the requirement of additivity in physical space **X**, we must distinguish between positions and velocities with the obvious notation

$$z = (x, v) \in \mathbf{Z} = \mathbf{X} \times \mathbf{V}, \qquad \mathbf{X} \subset \mathbb{R}^n, \quad \mathbf{V} \subset \mathbb{R}^n \tag{44}$$

For simplicity, we shall take $\mathbf{X} = \mathbb{R}^n$, $\mathbf{V} = \mathbb{R}^n$, but stress that such a requirement is not necessary for the analysis to follow. For any measurable set $\Delta \subset \mathbf{Z}$, we denote by $\pi\Delta$ its projection on to **X**, i.e.,

$$\pi\Delta = \{x \in \mathbf{X} : Z = (x, v) \in \Delta\} \tag{45}$$

We replace assumption (b) (7) by a weaker one:

(b′) If $f_1, f_2 \in \mathscr{D}(G)$ and $\pi \operatorname{supp} f_1 \cap \pi \operatorname{supp} f_2 = \varnothing$, then

$$G(f_1 + f_2) = G(f_1) + G(f_2) \tag{46}$$

To compensate partly for the weaker assumption (b′), we strengthen assumption (c) in a slight manner by:

(c′) All simple nonnegative functions with compact support (10) belongs to $\mathscr{D}(G)$.
Observe that (c′) follows automatically from (b) and (c) for functionals of class $\mathscr{A}$.

Definition 3. A functional $G(f)$: $L_p^{+,c} \to \mathbb{R}$ is called a functional of class $\mathscr{S}\mathscr{A}$ (semi-additive, or spatially additive) if $G(f)$ satisfies conditions (a), (b′), and (c′).

It is clear that (b′) is much weaker than (b) and there is no hope to show a generalization of Lemma 1 for this case. But it is remarkable that Theorem 1 remains valid for functionals of class $\mathscr{S}\mathscr{A}$ when the underlying dynamical systems are Newtonian (43). This is shown in the next section.

6. SEMI-ADDITIVE CONSERVATION LAWS

We consider the PDE (32) for the system (43)

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + w(x, v, t) \cdot \frac{\partial f}{\partial v} = 0 \tag{47}$$

with initial conditions

$$f|_{t=0} = f_0(z) = f_0(x, v), \qquad z = (x, v) \in \mathbf{Z} = \mathbf{X} \times \mathbf{V} \tag{48}$$

We assume that there exists a functional $G_t \in \mathscr{S}\mathscr{A}$ such that the following conservation law holds

$$G_t[f(\cdot, t)] = G_0(f_0) \tag{49}$$

Our goal is to show that equality (49) implies $G_t \in \mathscr{A}$, i.e., that any semi-additive conservation law is fully additive. Toward this end, we fix the function f_0 in (48) and *assume* that the conservation law (49) is also valid for the initial condition

$$f_0^{\Delta}(z) = f_0(z)\, \chi_{\Delta}(z) \tag{50}$$

where Δ is any bounded measurable set in $\mathbf{Z}$. To simplify notation, we express the solution of (43) in the form (29) and set

$$z_t = \hat{\mathbf{S}}(t)\, z = (x_t, v_t), \qquad z_{-t} = \hat{\mathbf{S}}^{-1}(t)\, z = (x_{-t}, v_{-t}) \tag{51}$$

Then $f(z, t) = f_0(z_{-t})$, and equality (49) for the initial condition (50) can be expressed as

$$G_t[(f_0 \chi_{\Delta}) \circ \hat{\mathbf{S}}^{-1}(t)] = G_0(f_0 \chi_{\Delta}) \tag{52}$$

since $[f_0 \chi_{\Delta} \circ \hat{\mathbf{S}}^{-1}(t)](z) = f_0(z_{-t})\, \chi_{\Delta}(z_{-t})$.

To show that semi-additivity of $G_t(\cdot)$ implies full (phase space) additivity for Newtonian systems we have to focus on the evolution of sets which are, and remain, disjoint in phase space, but whose projections on to physical space $\mathbf{X}$ do not. We denote by

$$\Delta = \Delta_1 \sqcup \Delta_2 \Leftrightarrow \Delta = \Delta_1 \cup \Delta_2 \qquad \text{if} \quad \Delta_1 \cap \Delta_2 = \varnothing \tag{53}$$

the union of two disjoint sets. In general, the notation

$$\Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_k = \bigsqcup_{j-1}^{k} \Delta_j$$

means the union of k pairwise disjoint sets $\Delta_1, \Delta_2, \ldots, \Delta_k$. Further, also denote

$$\Delta(t) = \hat{\mathbf{S}}(t)\,\Delta, \qquad \Delta(0) = \Delta \tag{54}$$

and observe that $\chi_\Delta(z_{-t}) = \chi_{\Delta(t)}(z)$.

Let us recall that (46) of property (b′) for the functional $G \in \mathscr{SA}$ means

$$G(f\chi_{\Delta_1 \sqcup \Delta_2}) = G(f\chi_{\Delta_1}) + G(f\chi_{\Delta_2}) \tag{55}$$

for any sets Δ_1 and Δ_2, with $\pi\Delta_1 \cap \pi\Delta_2 = \varnothing$, provided $f\chi_{\Delta_i} \in \mathscr{D}(g)$, $i = 1, 2$. We note that property (55) is valid for the functional G_t in (52) with any fixed t in the closed interval $[0, T]$, i.e., the interval on which the solution of (43) with initial conditions exists uniquely and where (30) holds.

For fixed f_0, we consider a function $\beta(\Delta; t; f_0)$ of the set Δ and time t, defined by

$$\beta(\Delta; t; f_0) = G_t\{[f_0 \circ \hat{\mathbf{S}}^{-1}(t)]\,\chi_\Delta\} \tag{56}$$

Then, from (52),

$$\beta(\Delta(t); t, f_0) = \beta(\Delta; 0; f_0), \qquad \Delta(t) = \hat{\mathbf{S}}(t)\,\Delta, \qquad \hat{\mathbf{S}}(0) = I \tag{57}$$

Moreover, for any $t \in [0, T]$, $\beta(\cdot; t; f_0)$ is semi-additive, i.e.,

$$\begin{gathered} \beta(\Delta_1 \sqcup \Delta_2; t; f_0) = \beta(\Delta_1; t; f_0) + \beta(\Delta_2; t; f_0) \\ \pi\Delta_1 \cap \pi\Delta_2 = \varnothing \end{gathered} \tag{58}$$

Conjecture. We assume that there exists $T = T(f_0) > 0$, such that $(f_0\chi_\Delta) \circ \hat{\mathbf{S}}^{-1}(t) \in \mathscr{D}(G_t)$ for any $0 \leqslant t \leqslant T$ and for any bounded measurable $\Delta \subset \mathbb{R}^{2n}$. Then the equalities (57), (58) imply the additivity of $\beta(\Delta; t; f_0)$, i.e.,

$$\beta(\Delta_1 \sqcup \Delta_2; t; f_0) = \beta(\Delta_1; t; f_0) + \beta(\Delta_2; t; f_0) \tag{59}$$

for any disjoint, bounded Δ_1 and Δ_2.

We now make some comments about the flow of the arguments used to prove this conjecture. First, we observe that it is enough to prove the conjecture for $\beta(\Delta; 0; f_0)$, since for any fixed $t \in [0, T]$, $\beta(\Delta; t; f_0) = \beta[\hat{\mathbf{S}}^{-1}(t)\,\Delta; 0; f_0]$ because of (52). So, if $\Delta = \Delta_1 \sqcup \Delta_2$,

$$\hat{\mathbf{S}}^{-1}(t)\,\Delta = (\hat{\mathbf{S}}^{-1}(t)\,\Delta_1) \sqcup (\hat{\mathbf{S}}^{-1}(t)\,\Delta_2)$$

and

$$\beta(\hat{\mathbf{S}}^{-1}(t)\,\Delta; 0; f_0) = \beta(\hat{\mathbf{S}}^{-1}(t)\,\Delta_1; 0; f_0) + \beta(\hat{\mathbf{S}}^{-1}(t)\,\Delta_2; 0; f_0)$$
$$= \beta(\Delta_1; t; f_0) + \beta(\Delta_2; t; f_0)$$

provided (59) is valid for $\beta(\Delta; 0; f_0)$. Here, we have used the fact that $\hat{\mathbf{S}}(t)$ is a one-to-one mapping of $\mathbf{Z}$ onto itself.

Let us choose now any pair of disjoint sets Δ_1 and Δ_2 and consider the conservation law (57) for $\Delta = \Delta_1 \sqcup \Delta_2$. If $\pi\Delta_1 \cap \pi\Delta_2 = \varnothing$, then (59) holds, owing to (58). Therefore, the interesting case is $\pi\Delta_1 \cap \pi\Delta_2 \neq \varnothing$, which we have depicted in Fig. 1 at some instant of time.

In this case, we can decompose Δ_1 and Δ_2 in the following manner:

$$\Delta_1 = \Delta_1^{(0)} \sqcup \Delta_1^{(1)}, \qquad \Delta_2 = \Delta_2^{(0)} \sqcup \Delta_2^{(1)}$$
$$\pi\Delta_1^{(0)} \cap \pi\Delta_2^{(0)} = \varnothing, \qquad \pi\Delta_1^{(1)} = \pi\Delta_2^{(1)} = \pi\Delta_1 \cap \pi\Delta_2$$

(The sets $\Delta_1^{(1)}$ and $\Delta_2^{(1)}$ are shaded as in the above figure).

Observe then, that

$$\beta(\Delta_1 \sqcup \Delta_2; f_0) = \beta(\Delta_1^{(0)}; f_0) + \beta(\Delta_2^{(0)}; f_0) + \beta(\Delta_1^{(1)} \sqcup \Delta_2^{(1)}; f_0)$$

because $\pi\Delta_1^{(1)} = \pi\Delta_2^{(1)}$, and we have denoted for the sake of brevity $\beta(\Delta; 0; f_0) \equiv \beta(\Delta; f_0)$. It suffices to prove the conjecture for the case $\pi\Delta_1 = \pi\Delta_2$ in (59). Toward this end, we introduce the concept of "T-separability."

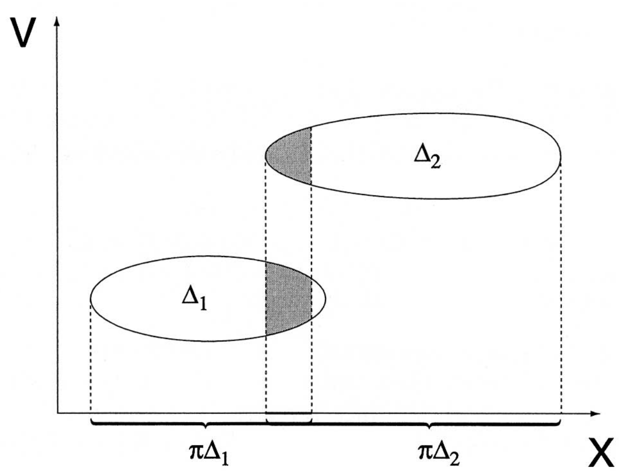

Fig. 1. The case $\pi\Delta_1 \cap \pi\Delta_2 \neq \varnothing$.

Definition 4 (Concept of T-separability). (A) Two disjoint sets Δ_1 and Δ_2 are called T-separable if there exists $t_* \in [0, T]$ such that $\pi\Delta_1(t_*) \cap \pi\Delta_2(t_*) = \varnothing$.

(B) Two disjoint sets Δ_1 and Δ_2 are called separable if for any (small) $T > 0$, there exists an integer N and a partition:

$$\Delta_i = \Delta_i^{(1)} \sqcup \Delta_i^{(2)} \sqcup \cdots \sqcup \Delta_i^{(N)}, \qquad i = 1, 2$$

such that all $2N$ subsets $\Delta_i^{(k)}$ are pairwise T-separable.

Then we can easily show

Lemma 3. We assume that there exists a partition

$$\Delta_i = \Delta_i^{(1)} \sqcup \Delta_i^{(2)} \sqcup \cdots \sqcup \Delta_i^{(N)}, \qquad i = 1, 2$$

such that:

(i) $\pi\Delta_1^{(k)} \cap \pi\Delta_1^{(l)} = \pi\Delta_2^{(k)} \cap \pi\Delta_2^{(l)} = \pi\Delta_1^{(k)} \cap \pi\Delta_2^{(l)} = \varnothing$, $k \neq l$;

(ii) Each pair $\Delta_1^{(k)}$, $\Delta_2^{(k)}$ $(k = 1, 2, ..., N)$ is T-separable.

Then

$$\beta(\Delta_1 \sqcup \Delta_2; f_0) = \beta(\Delta_1; f_0) + \beta(\Delta_2; f_0) \tag{60}$$

Proof. First, we consider the case when $N = 1$, i.e., we prove that (60) holds for any T-separable Δ_1 and Δ_2. By appealing to (57) and (58) at $t = t_*$, we obtain

$$\begin{aligned} \beta(\Delta_1 \sqcup \Delta_2; f_0) &= \beta(\Delta_1(t_*) \sqcup \Delta_2(t_*); t_*; f_0) \\ &= \beta[\Delta_1(t_*); t_*; f_0] + \beta[\Delta_2(t_*); t_*; f_0] \\ &= \beta(\Delta_1; f_0) + \beta(\Delta_2; f_0) \end{aligned}$$

Next, let us consider the general case $N > 1$. By using (i), we obtain

$$\beta(\Delta_1 \sqcup \Delta_2; f_0) = \sum_{k=1}^{N} \beta(\Delta_1^{(k)} \sqcup \Delta_2^{(k)}; f_0)$$

Since any pair $\Delta_1^{(k)}$ and $\Delta_2^{(k)}$ is T-separable by (ii), we can write

$$\beta(\Delta_1^{(k)} \sqcup \Delta_2^{(k)}; f_0) = \beta(\Delta_1^{(k)}; f_0) + \beta(\Delta_2^{(k)}; f_0)$$

and therefore the conclusion follows, since

$$\beta(\Delta_1 \sqcup \Delta_2; f_0) = \sum_{k=1}^{N} \beta(\Delta_1^{(k)}; f_0) + \sum_{k=1}^{N} \beta(\Delta_2^{(k)}; f_0)$$
$$= \beta(\Delta_1; f_0) + \beta(\Delta_2; f_0)$$

Remark. Note that each pair $\Delta_1^{(k)}$, $\Delta_2^{(k)}$ can have its own time of separation $t_*^{(k)}$; therefore it is quite possible that $\pi\Delta_1(t) \cap \pi\Delta_2(t) \neq \varnothing$ for all $t \in [0, T]$ in general.

Lemma 3 suggests a method of showing the conjecture. Without loss of generality, assume that Δ_i and $\Delta_i(t)$ are bounded in $\mathbb{R}^{2n}$ for $i = 1, 2$, and consider the case $\Delta = \Delta_1 \sqcup \Delta_2$, $\pi\Delta_1 = \pi\Delta_2$. Partition the set $\pi\Delta_1 = \pi\Delta_2$ in $\mathbb{R}^n$ into N disjoint subsets and consider the corresponding partition of Δ_1 and Δ_2 in $\mathbb{R}^{2n}$:

$$\Delta_i = \Delta_i^{(1)} \sqcup \cdots \sqcup \Delta_i^{(N)}, \quad i = 1, 2, \qquad \pi\Delta_1^{(k)} = \pi\Delta_2^{(k)} \tag{61}$$

It is clear that any such partition satisfies the first assumption of Lemma 3. The main problem now is to show that the second assumption (ii) of T-separability is automatically satisfied for sufficiently large N.

Lemma 4. For any compact Δ_1 and Δ_2, there exists a partition satisfying the conditions of Lemma 3, provided that for some $t_0 > 0$, there exists a unique solution of (43), $\hat{S}(t)\, z = z(t) \in C_1[0, t_0]$ for all $z \in \Delta_1 \sqcup \Delta_2$ and that $w(x, v, t) \equiv w(z, t) \in C((\Delta_1 \sqcup \Delta_2) \times [0, t_0])$ in (43).

Proof. Let us consider the aforementioned partition (61) into N subsets and denote by Ω_k, $1 \leqslant k \leqslant N$, $\Omega_k = \pi\Delta_1^{(k)} = \pi\Delta_2^{(k)}$, $d_N = \max_{1 \leqslant k \leqslant N} [diam(\Omega_k)]$. Since $\Omega := \pi\Delta_1 = \pi\Delta_2$ is bounded in $\mathbb{R}^n$, we can find sufficiently large N such that $d_N < \varepsilon$, $\varepsilon > 0$ and given.

We proceed with a contradiction argument: Assume for any partition (61), there is a pair $\Delta_1^{(k)}$ and $\Delta_2^{(k)}$ of subsets such that $\pi\Delta_1^{(k)}(t) \cap \pi\Delta_2^{(k)}(t) \neq \varnothing$ for all $t \in [0, T]$. Fix any $t \in (0, T]$, and consider a sequence of partitions such that $d_N \to 0$ for $N \to \infty$. Then for any $N = 1, 2,\ldots$, there exists a pair of phase points

$$z_i^{(N)}(t) = (x_i^{(N)}(t), v_i^{(N)}(t)) \in \Delta_i(t), \qquad i = 1, 2$$

such that

$$x_1^{(N)}(t) = x_2^{(N)}(t), \qquad |x_1^{(N)}(0) - x_2^{(N)}(0)| < d_N$$
$$(x_i^{(N)}(0), v^{(N)}(0)) \in \Delta_i$$

From the corresponding sequence $\{z_1^{(N)}(0), z_2^{(N)}(0)\}$, extract a convergent subsequence, and let $\{z_1^*(0), z_2^*(0)\}$ denote its limit. Now $z_i^*(0) \in \Delta_i$, $i = 1, 2$, since Δ_1 and Δ_2 are closed and $\Delta_1 \cap \Delta_2 = \emptyset$, and, moreover, $x_1^*(0) = x_2^*(0) := x^*(0) \in \Omega$, since $|x_1^{(N)}(0) - x_2^{(N)}(0)| \to 0$. Therefore, there exists $\delta > 0$ such that $|x_1 - x_2|^2 + |v_1 - v_2|^2 > \delta^2$ for any pair $(x_i, v_i) \in \Delta_i$, $i = 1, 2$. Therefore $|v_1^*(0) - v_2^*(0)| > \delta$.

Hence, for any given $t \in [0, T]$, we have found a pair of initial conditions for the Newtonian system (43)

$$x_1^*(0) = x_2^*(0); \qquad |v_1^*(0) - v_2^*(0)| > \delta \tag{62}$$

such that $x_1^*(t) = x_2^*(t)$ for the corresponding solutions. On the other hand let us consider any two phase space trajectories $z_i(t) = \hat{\mathbf{S}}(t)\, z_i$, $z_i \in \Delta_i$, $i = 1, 2$, and $x_1 = x_2$. For each trajectory, we denote

$$w_i(t) = dv_i(t)/dt = w(x_i(t), v_i(t), t), \qquad i = 1, 2$$

from which we obtain

$$x_i(t) = x_i + v_i t + \int_0^t d\tau(t - \tau)\, w_i(\tau), \qquad i = 1, 2$$

Now, if $x_1(t_1) = x_2(t_1)$, then we obtain

$$v_2 - v_1 = t_1^{-1} \int_0^{t_1} d\tau(t_1 - \tau)[w_1(\tau) - w_2(\tau)]$$

from which the following estimate results

$$\delta \leqslant |v_1 - v_2| \leqslant t_1^{-1} \int_0^{t_1} d\tau(t_1 - \tau)[|w_1(\tau)| + |w_2(\tau)|] \tag{63}$$

The estimate (63) enables us to apply our contradiction argument to complete the proof of Lemma 4. We have assumed that $w(z, t)$ in (43) is continuous. Therefore, the acceleration $(dv/dt) = w[\hat{\mathbf{S}}(t)\, z, t]$ is uniformly bounded on $(\Delta_1 \cup \Delta_2) \times [0, t_0]$ by

$$\|w\| := \max\{|w(\hat{\mathbf{S}}(t)\, z, t)| : z \in \Delta_1 \cup \Delta_2,\ t \in [0, t_0]\}$$

Thus, if $t \leqslant t_1$ in (63), we have

$$\delta \leqslant 2t_1^{-1}\, \|w\| \int_0^{t_1} d\tau(t_1 - \tau) = t_1\, \|w\|$$

from which we obtain an estimate of t_1, i.e.,

$$t_1 \geqslant \min(t_0, \delta/\|w\|) \tag{64}$$

for the time t_1 at which $x_1(t_1) = x_2(t_1)$ for any two trajectories $z_i(t) = \hat{\mathbf{S}}(t)\, z_i$, $z_i \in \Delta_i$, $i = 1, 2$, with $x_1 = x_2$. But, in assuming that the conclusion of Lemma 4 is false, we obtained in (62) initial conditions for which (43) yields a solution such that $x_1^*(t) = x_2^*(t)$ for any small t. But this contradicts inequality (64), and the proof of Lemma 4 is complete.

Hence, we can apply Lemma 3 to any pair of disjoint closed bounded sets Δ_1 and Δ_2. However, we have assumed that our functionals $G_t \in \mathscr{S}\mathscr{A}$, $0 \leqslant t \leqslant T$, are continuous in the L_p-norm for some $p \in [1, \infty)$, because of assumption (a) at the beginning of Section 2. Therefore $\beta(\Delta; t)$ (56) depends continuously on $\chi_\Delta(\cdot)$, $\Delta \in \mathbf{Z}$, and (59) is valid for any pair of disjoint and bounded sets Δ_1 and Δ_2. With this remark, the proof of the conjecture is complete.

Corollary. If $f_0 = f_1 + f_2$ satisfies the conditions of the conjecture and $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \varnothing$, then

$$G_0(f_1 + f_2) = G_0(f_1) + G_0(f_2)$$

Proof. Denote $\Omega_i = \operatorname{supp} f_i$, $i = 1, 2$, with Ω_i bounded and measurable in $\mathbb{R}^{2n}$. Observe that $f_1, f_2 \in \mathscr{D}(G_0)$, since $f_i = f_0 \chi_{\Omega_i}$. Then we simply put $f_0 = f_1 + f_2$, $t = 0$ with $\Delta_i = \Omega_i$ $(i = 1, 2)$ in (56) and (59), and the proof is complete.

Finally, let us specify a set of initial data $f_0(z)$ satisfying the conditions of the conjecture. We still have not used assumption (c′) that $\mathscr{D}(G_t)$ contains all simple functions (10). Observe that if $f_0(z)$ is a simple function then $f_0(z_{-t})\, \chi_\Delta(z_{-t})$ is also simple for any $0 \leqslant t \leqslant T$, $\Delta \subset \mathbb{R}^{2n}$. It is quite natural to assume that if the conservation law (49) is valid on a certain time interval $0 \leqslant t \leqslant T$ for all nonnegative simple functions $f_0(z)$ with compact support, then the conjecture is also valid for such functions. Moreover, we can easily prove now that

$$G_t(f_1 + f_2) = G_t(f_1) + G_t(f_2), \qquad 0 \leqslant t \leqslant T \tag{65}$$

for any simple f_1 and f_2 if $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \varnothing$. It is enough to note that

$$G_t[f_1 + f_2] = G_0[f_1 \circ \hat{\mathbf{S}}(t) + f_2 \circ \hat{\mathbf{S}}(t)]$$

and that $f_i \circ \hat{\mathbf{S}}(t)$, $i=1, 2$, are also simple functions with disjoint supports. Therefore, we can apply the above Corollary to the functions $f_i \circ \hat{\mathbf{S}}(t)$, $i=1, 2$, and get

$$\begin{aligned} G_0[f_1 \circ \hat{\mathbf{S}}(t) + f_2 \circ \hat{\mathbf{S}}(t)] &= G_0(f_1 \circ \hat{\mathbf{S}}(t)) + G_0(f_2 \circ \hat{\mathbf{S}}(t)) \\ &= G_t(f_1) + G_t(f_2) \end{aligned}$$

thereby showing (65).

Remark. The only property of simple functions needed for the proof of (65) is the assumption that (49) is fulfilled for any simple $f_0(z)$.

Our functionals are continuous in $L_p^{+,c}$; therefore (65) implies full aditivity not only for simple $f_1(z)$ and $f_2(z)$, but for all $f_i \in \mathscr{D}(G_t)$, $i=1, 2$. We have shown

Theorem 2. For Newtonian systems, with continuous $w(x, v, t)$, we can replace the requirement of phase space additivity by semi-additivity or physical space additivity.

In the next section, we extend our results to a wider class of Hamiltonian and Lagrangian systems.

7. LAGRANGIAN AND HAMILTONIAN SYSTEMS

We consider a Lagrangian system with Lagrangian $L(x, \dot{x}, t)$, $x \in \mathbb{R}^n$, $\dot{x} = dx/dt \in \mathbb{R}^n$, $0 \leqslant t \leqslant T$. Then the Euler–Lagrange equations yield

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{66}$$

If we denote $v = \dot{x}$, $L = L(x, v, t)$, then we formally obtain

$$\frac{dx_i}{dt} = v_i \tag{67}$$

$$\frac{\partial^2 L}{\partial v_i \partial t} + \frac{\partial^2 L}{\partial v_i \partial v_j} v_j + \frac{\partial^2 L}{\partial v_i \partial v_j}\frac{dv_j}{dt} = \frac{\partial L}{\partial x_i} \tag{68}$$

where a summation over the running index $j = 1, 2, \ldots, n$ is assumed. We introduce a matrix $\hat{\mathbf{M}}$ such that

$$[\hat{\mathbf{M}}^{-1}(x, v)]_{i,j} = \frac{\partial^2 L}{\partial v_i \partial v_j}, \qquad i, j = 1, \ldots, n \tag{69}$$

and assume a nondegeneracy condition, i.e., that

$$\det\left(\frac{\partial^2 L}{\partial v_i\, \partial v_j}\right) \neq 0 \tag{70}$$

Then the Euler–Lagrange equations can be expressed as the Newtonian system of two equations

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = w(x, v, t) := \hat{\mathbf{M}}\left[\frac{\partial L}{\partial x} - \left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x}\right)\frac{\partial L}{\partial v}\right] \tag{71}$$

Lemma 5. Theorem 2 is valid for the Euler–Lagrange equation (67)–(68) (expressed in the variables x and v (67)—(68)) if

(i) The partial derivatives $L_x, L_{xt}, L_{xv}, L_{vv}$ of the Lagrangian $L(x, v, t)$ are continuous on $\mathbb{R}^n \times \mathbb{R}^n \times [0, T]$;

(ii) The Lagrangian is nondegenerate, i.e., (70) is valid for all $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$, $t \in [0, T]$.

Proof. The proof follows easily when we observe that $w(x, v, t)$ in (71) is a continuous function under assumptions (i) and (ii).

If we consider nondegenerate Lagrangian systems (66) and (70), it is always convenient to pass to the Hamiltonian formulation, since the phase volume is preserved in (x, p) space. Therefore, denote, as is customary,

$$p = \frac{\partial L}{\partial v}, \qquad H(x, p, t) := v\frac{\partial L}{\partial v} - L \tag{72}$$

and observe that the Hamiltonian is also nondegenerate when (70) holds

$$\det\left(\frac{\partial^2 H}{\partial p_i\, \partial p_j}\right) \neq 0 \tag{73}$$

Then we can obtain Hamiltonian equations for the spatial and momentum variables (x, p),

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}, \qquad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n \tag{74}$$

and the corresponding Liouville equation for a distribution function $f(x, v, t)$ is

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial f}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial p} = 0, \qquad f|_{t=0} = f_0(x, p) \tag{75}$$

Observe that the momentum p does not coincide now with the velocity v and therefore we cannot apply directly Theorem 2 for Newtonian systems. But condition (73) shows that at any $x \in \mathbb{R}^n$, $t \in [0, T]$, there is a one-to-one correspondence between the p and v variables. Therefore we can easily transform the system (98) to its original Newtonian–Lagrangian form (74), apply Theorem 2, and formulate the final result in (x, p)-variables afterwards.

We denote $z = (x, p)$ and consider functionals $G(f)$ acting on the z-variable. As usual, our phase space is $\mathbf{Z} = \mathbb{R}^n \times \mathbb{R}^n$. We recall that the functional G is called semi-additive (i.e., $G \in \mathscr{S}\mathscr{A}$) if it is additive only on the x-variable. The notation $z = \hat{\mathbf{S}}(t)\, z_0$ is used for the solution of (74) with initial condition $z(0) = z_0$, while $z_0 = \hat{\mathbf{S}}^{-1}(t)\, z$ denotes an inverse transformation.

We can easily show the following

Theorem 3. Assume that the Hamiltonian $H(x, p, t)$ in (74) satisfies the conditions:

(i) The partial derivatives H_x, H_{pt}, H_{px}, H_{pp} are continuous on $\mathbb{R}^n \times \mathbb{R}^n \times [0, T]$;

(ii) $H(x, p, t)$ is nondegenerate, i.e., (73) is valid for all $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $t \in [0, T]$.

Then any semi-additive conservation law (75), expressed by

$$\frac{d}{dt} G_t(f) = 0 \tag{76}$$

with $G_t \in \mathscr{S}\mathscr{A}$ for any $t \in [0, T]$, is completely defined by a certain function $g(u; z)$ of $(2n+1)$ variables $(u \geqslant 0, z \in \mathbb{R}^n \times \mathbb{R}^n)$. The kernel $g_t(u; z)$ of G_t can be written as

$$g_t(u; z) = g(u; \hat{\mathbf{S}}^{-1}(t)\, z) \tag{77}$$

To carry out a proof, we note that conditions (i) and (ii) are sufficient for transforming (74) to Newtonian form and applying Theorems 1 and 2. The final step in the proof is to use the Liouville theorem, $D[\hat{\mathbf{S}}(t)\, z]/D(z) \equiv 1$; therefore, we obtain (77) from (40). We do not need to check differentiability of G_t, since (76) is simply equivalent to $G_t(f) = const.$ With this observation, we prove Theorem 3.

It is well known that the substitution $x = p'$, $p = -x'$, $H(x, p, t) = H'(x', p', t) = H(p', -x', t)$ results in a new Hamiltonian system:

$$\frac{dx'}{dt} = \frac{\partial H'}{\partial p'}, \qquad \frac{dp'}{dt} = -\frac{\partial H'}{\partial p'} \tag{78}$$

Formally, there is not a big difference between the variables x and p. But how can we distinguish between them on the formal mathematical level to be able to apply the preceding theorem?

The answer is as follows. A "true" space variable is distinguished by condition (70) defining the non-degeneracy of $H(x, p, t)$. If, however, the similar condition

$$\det\left(\frac{\partial^2 H}{\partial x_i\,\partial x_j}\right) \neq 0 \tag{79}$$

is satisfied analogously for the p-variable, then Theorem 3 will be applicable for functionals being "additive in momentum space." To prove this, it suffices to consider the transformed Hamiltonian system (78).

It is clear, for example, that for the harmonic oscillator,

$$H = \tfrac{1}{2}(|p|^2 + |x|^2) \tag{80}$$

one can interpret the term "semi-additive" (or, $G_t \in \mathscr{S}\mathscr{A}$) in Theorem 3 in two different versions: (i) Additivity only in the spatial x-variable; (ii) Additivity only in the spatial p-variable. This interpretation applies in all cases when both (70) and (79) are satisfied.

With the typical Hamiltonian of classical mechanics,

$$H(x, v, t) = \tfrac{1}{2}\,|p|^2 + V(x, t)$$

where $V(x, t)$ denotes a potential, we are always sure that (70) is satisfied, whereas the second condition (79) is valid only in special cases. Thus, the two canonical variables x and p are clearly separated in the majority of physically relevant problems.

8. GENERALIZATIONS

Let us consider a general case of the dynamical system in $\mathbb{R}^{2n}$:

$$\frac{dz}{dt} = v(z, t), \qquad z \in \mathbb{R}^{2n}, \quad 0 \leqslant t \leqslant T \tag{81}$$

(where we do not know in advance the variables playing the role of a space variable $x \in \mathbb{R}^n$). A practical way to use Theorem 2 in this case is to seek a global transformation (with range in $\mathbb{R}^n \times \mathbb{R}^n \times [0, T]$) defined by

$$x = \Phi(z, t), \qquad v = \Psi(z, t), \qquad x \in \mathbb{R}^n, \qquad v \in \mathbb{R}^n$$

of (81) to the Newtonian system (43). If such a transformation does exist, then we can apply Theorem 2 to the transformed system and then reformulate the result in terms of the previous dynamical system (81). It is clear in advance that such transformations are not unique in the general case, the harmonic oscillator (80) being the simplest example. In such cases, there are several sets of variables for which the condition of semi-additivity (i.e., additivity in $\mathbb{R}^n$) implies the condition of full additivity in $\mathbb{R}^{2n}$ for conservation laws (Theorem 2).

Moreover, we observe that the Newtonian system (43) is obviously equivalent to the second order ODE in $\mathbb{R}^n$

$$\frac{d^2x}{dt^2} = w\left(x, \frac{dx}{dt}, t\right), \qquad x \in \mathbb{R}^n \tag{82}$$

and this explains on an intuitive level the sufficiency of additivity in $\mathbb{R}^n$ for additivity in $\mathbb{R}^{2n}$ for conserved functionals.

Let us now consider an equation of Nth order in $\mathbb{R}^n$ ($N = 2, 3,...$)

$$\begin{gathered} \frac{d^N x}{dt^N} = F(x, x^{(1)},..., x^{(N-1)}; t), \qquad x^{(k)} = \frac{d^k x}{dt^k} \\ k = 1, 2,..., N-1, \qquad x \in \mathbb{R}^N, \qquad t \in [0, T] \end{gathered} \tag{83}$$

with initial conditions

$$x(0) = x^{(0)}, \qquad x^{(k)}(0) = y_k^{(0)}, \qquad k = 1,..., N-1 \tag{84}$$

This equation reduces to the usual dynamical system (first order ODE) in $(\mathbb{R}^n)^N$ by the obvious substitution:

$$x^{(k)} = y_k, \qquad k = 1, 2,..., N-1; \qquad y_k \in \mathbb{R}^n \tag{85}$$

Our phase coordinate now is

$$z = (x, y_1,..., y_{N-1}) \in \mathbb{R}^{Nn} \tag{86}$$

while $x \in \mathbb{R}^n$ is still regarded as a space variable. The dynamical system yields

$$\begin{gathered} \frac{dx}{dt} = y_1, \qquad \frac{dy_k}{dt} = y_{k+1}, \qquad k = 1, 2,..., N-2 \\ \frac{dy_{N-1}}{dt} = F(x, y_1,..., y_{N-1}; t) \end{gathered} \tag{87}$$

We can obviously apply Theorem 1 to this system. Moreover, we can try to extend the results of Section 6 (Theorem 2) to this case. The phase space $\mathbf{Z} = \mathbb{R}^{Nn}$ can now be expressed as a Cartesian product $\mathbf{Z} = \mathbf{X} \times \mathbf{Y}$, $\mathbf{X} = \mathbb{R}^n$, $\mathbf{Y} = \mathbb{R}^{(N-1)n}$, and we are in a position to repeat the arguments of Section 6, with the obvious minor alterations.

Toward this end, we have to consider the distribution function $f(x, y_1,..., y_{N-1}; t)$ satisfying the equation

$$\frac{\partial f}{\partial t} + y_1 \frac{\partial f}{\partial x} + \sum_{k=1}^{N-2} y_{k+1} \frac{\partial f}{\partial y_k} + F \frac{\partial f}{\partial y_{N-1}} = 0 \tag{88}$$

with an initial condition $f|_{t=0} = f_0(z)$, $z = (x, y_1,..., y_{N-1})$ (Compare with (27) and (28)). Thus our problem is to describe all conservation laws for (88) which are expressed as

$$G_t[f(\cdot; t)] = G_0(f_0) \tag{89}$$

Moreover, for each fixed $t \in [0, T]$, these functionals G_t are assumed to be additive in $\mathbb{R}^n$ with respect to the x-variable. Thus, the notion of semi-additivity introduced in Definition 3 in Section 5 is extended to functionals $G_t \in \mathscr{S}\mathscr{A}$ for the case $\mathbf{Z} = \mathbf{X} \times \mathbf{Y}$ under consideration. Our goal now is to show that the theorems (corresponding to $N = 2$ in (87)) can be easily generalized to arbitrary $N = 2, 3,...$.

All considerations from Section 6 can be repeated for the general case $N \geqslant 2$ almost without changes, The only point we need to clarify is the second half of the proof to Lemma 4, where the properties of the Newtonian system (43) were exploited. To extend the contradiction argument in the proof of Lemma 4 for Newtonian systems to the setting here, we must show:

Lemma 6. Let Δ_1 and Δ_2 be two compact disjoint measurable sets in $\mathbb{R}^{Nn}$. Let $z_i(t) \in C_1[0, t_0]$, $i = 1, 2$, be the solutions of (87) with initial conditions $z_i(0) \in \Delta_i$, $i = 1, 2$, such that $x_i(0) = x_2(0)$. Then $x_1(t) = x_2(t)$ is impossible for sufficiently small $t > 0$ if $F(z, t) \in C\{(\Delta_1 \cup \Delta_2) \times [0, t_0]\}$.

Proof. To be more precise, we denote

$$z_i(t) = \{x_i(t), \mathbf{y}_i(t)\}, \qquad i = 1, 2,$$

$$\mathbf{y}_i(t) = (y_1^{(i)}(t), y_2^{(i)}(t),..., y_{N-1}^{(i)}(t)) \in \mathbb{R}^{(N-1)n}$$

and observe that $x_i(t) \in C_N[0, t_0]$ if the assumptions of Lemma 6 are fulfilled. Moreover,

$$y_k^{(i)}(0) = \left.\frac{d^k x}{dt^k}\right|_{t=0}, \qquad k = 1,..., N-1; \quad i = 1, 2$$

and

$$\left|\frac{d^N x_i}{dt^N}\right| = |F(z, t)| \leqslant \|F\| < \infty, \qquad 0 \leqslant t \leqslant t_0$$

A Taylor series expansion for $x_i(t)$ yields

$$x_1(t) - x_2(t) = \sum_{k=1}^{N-1} [y_k^{(1)}(0) - y_k^{(2)}(0)]\, t^k/k! + \Psi(\theta t)\, t^N/N!$$

where $0 \leqslant \Theta \leqslant 1$ and

$$\Psi(t) = \frac{d^N}{dt^N} [x_1(t) - x_2(t)]$$

If, at a certain $t > 0$, $x_1(t) = x_2(t)$, then

$$\frac{\Psi(\Theta t)}{N!} = t^{-N} \sum_{k=1}^{N-1} a_k \frac{t^k}{k!}, \qquad a_k = y_k^{(1)}(0) - y_k^{(2)}(0)$$

We have assumed that Δ_1 and Δ_2 are compact in $\mathbb{R}^{Nn}$ and are disjoint. Thus,

$$\sum_{k=1}^{N-1} a_k^2 > \delta \tag{90}$$

for certain $\delta > 0$, since $x_1(0) = x_2(0)$. On the other hand, $|\Psi(\theta t)| \leqslant 2\,\|F\| < \infty$. So

$$t^{-N} \left| \sum_{k=1}^{N-1} a_k t^k/k! \right| \leqslant 2\,\|F\|/N! < \infty \tag{91}$$

To conclude the proof, let $a_1 = a_2 = \cdots = a_{l-1} = 0$, $a_l \neq 0$, $(l \leqslant N-1)$. Then for $t \to 0$, we obtain

$$|a_l| + o(t) \leqslant 2l!\ \|F\|\ t^{N-l}/N!$$

which contradicts (90) for sufficiently small $t>0$. Hence, (91) can be satisfied only for $t \geqslant t_* > 0$, where t_* depends only on $\|F\|$ and the sets Δ_1 and Δ_2. This completes the proof.

Thus, Lemma 4 is valid in the wider context for the system (87), which is equivalent to the Nth order equation (83). All other considerations from Section 6 remain the same if we replace the Newtonian system (43) by the system (87). With these remarks, we have the following generalization of Theorem 2.

Theorem 4. For the dynamical system (87) with continuous function $F(z, t)$ and arbitrary $N = 2, 3, \ldots$, we can replace assumption $G_t \in \mathscr{A}$ of Theorem 1 (with $m = Nn$) by a weaker assumption $G_t \in \mathscr{SA}$ (i.e., additivity with respect to $x \in \mathbb{R}^n$).

A more general question can be asked: What will occur if we consider a general dynamical system for $z = (x, y) \in \mathbb{R}^{n+s}$,

$$\frac{dx}{dt} = v(z, t), \qquad \frac{dy}{dt} = w(z, t), \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^s \tag{92}$$

with semi-additivity (with respect to $x \in \mathbb{R}^n$) imposed for conservation laws $G_t[f(\cdot, \cdot, t)] = const.$ on the solution of

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + w \cdot \frac{\partial f}{\partial y} = 0? \tag{93}$$

In what cases, can we prove a result analogous to Theorems 1 and 2? It is clear from the above consideration that a key point is Lemma 4 from Section 6 which must be extended to show Lemma 6 for the setting described by (92) and (93). Without additional information about $v(z, t)$ and $w(z, t)$ in (92) we cannot obtain a result analogous to Lemma 6. Indeed, for (92) and (93), we can construct counterexamples showing that the conclusion of Lemma 6 is false in general. Therefore, one can use Lemma 6 as a sufficient condition showing for what cases a stronger version of Theorem 1 is valid, which we state below.

Theorem 5. With $m = n + s$, and $G_t \in \mathscr{SA}$, Theorem 1 is valid for any dynamical system (92) if the following condition is fulfilled: For any disjoint compact sets $\Delta_1, \Delta_2 \in \mathbb{R}^{n+s}$, and for any two solutions $z_i(t)$, $i = 1, 2$, satisfying the initial conditions

$$z_i(0) = (x_i^{(0)}, y_i^{(0)}) \in \Delta_i, \qquad x_1^{(0)} = x_2^{(0)}, \qquad i = 1, 2 \tag{94}$$

there exists $t_*(\Delta_1, \Delta_2) > 0$ such that $|x_1(t) - x_2(t)| > 0$ for all $0 < t < t_*(\Delta_1, \Delta_2)$.

In particular, this condition is fulfilled if $s = (N-1)n$ in (92) and if for any fixed $x \in \mathbb{R}^n$ and $t \in [0, T]$, there exists an appropriate change of variables $\{y_1, ..., y_s\} \to \{\tilde{y}_1, ..., \tilde{y}_s\}$ such that the resulting system for $\tilde{z} = (x, \tilde{y})$ is similar in structure to (87).

If all spatial trajectories $x(t)$ of the dynamical system (92) are analytic for $|t| < \varepsilon$, then the condition of Theorem 5 is violated if and only if there exist two identical spatial trajectories $x_1(t) = x_2(t)$, $|t| < \varepsilon$, such that $y_1^{(0)} \neq y_2^{(0)}$.

Remark. For illustration, we mention a trivial example for which the condition of Theorem 5 is not fulfilled. We consider (92) for the case $n = s = 1$, $v = w = x$. In this case, the solution of (93) with initial data $f|_{t=0} = f_0(x, y)$ can be expressed as

$$f(x, y, t) = f_0(xe^{-t}, y - x(1 - e^{-t}))$$

and the functional

$$G_t(f) = \int_{-\infty}^{\infty} dx\, e^{-t} \left[\int_{-\infty}^{\infty} dy\, f(x, y) \right]^2, \qquad G_t[f(\cdot, \cdot, t)] = const$$

yields an example of a semi-additive, but not fully additive conservation law.

Finally, we discuss a possible generalization of our class of functionals, described in Section 2. The kernel $g(u; z)$ in (8) may perhaps be defined only for $a \leqslant u \leqslant b$ as for example $g = (1 - u^2)^{1/2}$, $a = -b = 1$, or for all $u \in \mathbb{R}$. In such cases, one can replace the condition of nonnegativity, $f(z) \geqslant 0$ by $a \leqslant f(z) \leqslant b$, or eliminate this condition altogether. It is clear that we have never used the condition $f(z) \geqslant 0$ in the preceding deliberations, and hence all results remain valid for any similar restriction on $f(z)$. For Theorem 2, we can relax the requirement of nonnegativity in (c') by requiring that simple functions with values between a and b are in $\mathscr{D}(G_t)$, $a < b$. Then Theorem 2 would be valid in the corresponding class of functions $f(z, t)$, as $f(z, t)$ has the same values for $t > 0$ as does $f(z, 0)$.

There is, however, a more interesting generalization of the set of functions which is important in applications. Until now, we have considered the case of functions $f(z)$ with compact support. However, an important solution of the Liouville equation (75) with time-independent Hamiltonian $H(x, p)$ is the Gibbs distribution

$$f_*(x, p) = A \exp[-BH(x, p)] \tag{95}$$

with positive constants A and B. In the general case of an autonomous dynamical system

$$\frac{dz}{dt} = v(z), \qquad div_z v = 0$$

we can assume that the corresponding Liouville equation

$$\frac{\partial f}{\partial t} + v(z) \cdot \frac{\partial f}{\partial z} = 0 \tag{96}$$

has a stationary solution

$$f_*(z) = \omega(z), \qquad \omega(z_t) = \omega(z)$$

Let us consider now a class of solutions of (96) satisfying other conditions at infinity and

$$\lim_{|z| \to \infty} |f(z, t) - \omega(z)| = 0 \tag{97}$$

and proceed to generalize all results obtained previously to this case. Towards this end, we can simply put

$$f(z, t) = \omega(z) + f'(z, t), \qquad f' \to 0 \qquad \text{as} \quad |z| \to \infty$$

and observe $f'(z, t)$ also satisfies (96). It is clear that for fixed $\omega(z)$, any functional $\Gamma(f)$ reduces to a certain functional $G(f')$ by

$$\Gamma(f) = G(f - \omega)$$

From Theorem 1, we know that any conservation law for $G_t \in \mathscr{A}$ can be expressed (at least for simple $f'(z, 0)$) as

$$G_t[f'(\cdot, t)] = \int dz \; g[f'(z_{-t}, 0); z_{-t}] = const$$

or

$$\Gamma[f(\cdot, t)] = \int dz \{\gamma[f(z_{-t}, 0); z_{-t}] - \gamma[\omega(z); z_{-t}]\}$$

where a new kernel $\gamma(u; z)$ is related to $g(u; z)$ by the equality (with $\omega(z_{-t}) = \omega(z)$)

$$\gamma[u + \omega(z); z] - \gamma[\omega(z); z] = g(u; z)$$

Using these formulas, one can easily reformulate all our results in terms of distribution functions $f(z, t)$ satisfying the boundary condition (97).

Therefore, all the results of Sections 2–7 can be significantly generalized in different directions. In forthcoming work, we shall consider possible applications of these results to classical nonlinear kinetic equations. In particular, the results in this work will make it possible to describe clearly the most general class of functionals on solutions of the Boltzmann equation, with external fields, in which the H-functional and conservation laws are defined uniquely.

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